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Generators of Positive C_0-semigroups on Banach Lattices

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Let $\{T_t\}_{t\geq 0}$ be a C_0-semigroup of positive operators on a real Banach lattice E with generator A. In this paper, by the use of Yosida approximation, it is shown that if E is σ-order complete, then A satisfies abstract Kato's inequality for $u \in D(A)$ and $f \in D(A^+) \cap E^*_+$:

$$\langle |u|, A^+ f\rangle \geq \langle (\mathrm{sgn}\, u)Au, f\rangle ,$$

where A^+ denotes the generator of the dual semigroup of $\{T_t\}_{t\geq 0}$. In the special case in which $E = L^p(\mathbb{R}^n)$ $(1\leq p<\infty)$ and A is a differential operator with smooth coefficients, it is proved that the order of A is at most 2 and the principal part of A is (degenerate) elliptic, provided $C_0^\infty(\mathbb{R}^n)$ is a core of A. The paper contains an improvement of a recent result of Baoswan Wong-Dzung on the generation of positive semigroups by degenerate elliptic second order differential operators.

§1. Introduction

In this paper we deal with the problem of characterization of positive (some authors prefer the term "positivity preserving") C_0-semigroups on Banach lattices through their (infinitesimal) generators.

Let $\{T_t\}_{t\geq 0}$ be a C_0-semigroup on a Banach lattice E with generator A. Then $\{T_t\}_{t\geq 0}$ is said to be positive if T_t is positive for any $t \geq 0$, namely $T_t x \geq 0$ whenever $x \geq 0$. The resolvent $(\lambda - A)^{-1}$ of A is denoted by $R(\lambda, A)$. Then the equalities

$$T_t = \text{s-}\lim_{n\to\infty} [(n/t)R((n/t),A)]^n \quad (t > 0),$$

and

$$R(\lambda,A) = \int_0^\infty e^{-\lambda t}\, T_t\, dt \quad \text{(for sufficiently large } \lambda \text{)}$$

immediately imply that $\{T_t\}_{t\geq 0}$ is positive if and only if $R(\lambda,A)$ is positive for sufficiently large λ. (Concerning this point, we note that Greiner-Voigt-Wolff [5] proved the positivity of $R(\lambda,A)$ for $\lambda > \sup\{\operatorname{Re}\mu \ ;\ \mu \in \sigma(A)\}$ in case $\{T_t\}_{t\geq 0}$ is positive, where $\sigma(A)$ is the spectrum of A.) Thus we get the solution to our problem on the "resolvent level". But in practice, the resolvent $R(\lambda,A)$ is rather distant from A itself. So we want to obtain a characterization which is more closely related to the generators. Such a goal has been sought by several mathematicians. Phillips [14] seems to be the first result in this direction, in which C_0-semigroups of positive contractions on Banach lattices are characterized by the dispersiveness of their generators. Related results were obtained, for example, by Hasegawa [6] and Sato[15].

Recently this type of characterization has been brought to a culmination by the paper of Arendt-Chernoff-Kato [2]. §2 consists of an exposition of their results and some remarks, by which we try to put some earlier results in the light of their theory and to prepare for later sections.

The paper of Nagel-Uhlig [11] opened an entirely new way to approach this problem. They showed that the abstract "Kato's inequality", which was first introduced to deal with the self-adjointness problem of Schrödinger operators, is relevant to the characterization of positive C_0-semigroups. In §3 , we prove in an elementary fashion that the generator of a positive C_0-semigroup on an σ-order complete Banach lattice satisfies the abstract Kato's inequality. Some related results are also treated in this section.

§4 is devoted to show that the differential operators satisfying the abstract Kato's inequality must be of order at most 2.

In the final §5, we improve the result in Baoswan Wong-Dzung [19] on generation of positive semigroups by degenerate elliptic second order differential operators. Our method depends on the perturbation theory obtained by one of the authors.

Throughout the paper we freely use the standard notations concerning Banach lattices, for which the reader is referred to Schaefer[17]. In this paper we exclusively work with real Banach lattices, since positive C_0-semigroups and their generators are necessarily real operators.

Remark 1.1. After the completion of the manuscript, the authors learned that Arendt [1] gave a simple proof of Theorem 3.3. of this paper. Nevertheless the authors think that the use of Yosida approximation in the theory of positive semigroups is still of value, as the proof of Proposition 3.6 shows.

§2. Dispersive operators and positive semigroups

W.Arendt, P.R.Chernoff and T.Kato based their theory in [2] on the notion of half-norms. Recall that a functional Φ on a Banach space X is called a half-norm on X if Φ is a positively homogeneous subadditive functional satisfying $\Phi(x)+\Phi(-x) > 0$ for any $x \neq 0$. A typical example of a half-norm is the canonical half-norm on an ordered Banach space E : namely we define $\Phi(x) = d(-x,E_+)$, where $d(-x,E_+)$ denotes the distance from $-x$ to E_+(=the positive cone of E). In particular if E is a Banach lattice, the canonical half-norm on E is given by $\Phi(x) = \|x^+\|$. This is a consequence of the following observations: $d(-x,E_+) \leqq \|-x - x^-\| = \|x^+\|$;

For $y \in E_+$,$(-x - y)^- \geq (-x)^- = x^+$, hence $\|-x - y\| \geq \|x^+\|$. (Thus we have shown that x^- is a nearest point in E_+ to $-x$.)

To obtain fruitful results, we always assume that Φ is continuous with respect to the norm topology.

Let A be a linear operator with domain $D(A)$ and range $R(A)$ in X and let Φ be a half-norm on X . Then A is said to be Φ-dissipative if $\Phi(x - \mu Ax) \geq \Phi(x)$ holds for any $x \in D(A)$ and $\mu > 0$. When A further satisfies the condition $R(I- \mu A) = X$ for any $\mu > 0$, A is called an m-Φ-dissipative operator. If a linear operator A in X has the property that $A - \alpha$ is Φ-dissipative [resp. m-Φ-dissipative] for some constant α , A is said to be quasi-Φ-dissipative [resp.quasi-m-Φ-dissipative]. By Theorem 2.4 in [2], a densely defined quasi-Φ-dissipative operator A is closable and its closure $\tilde{A}$ is also quasi-Φ-dissipative. We call a densely defined linear operator A essentially quasi-m-Φ-dissipative if its closure $\tilde{A}$ is quasi-m-Φ-dissipative.

If the canonical half-norm on a Banach lattice is concerned, the term " Φ-dissipative" is replaced by "dispersive". So we say "essentially m-dispersive" instead of "essentially m-Φ-dissipative", for example.

On the other hand, an everywhere defined linear operator T on X is called a Φ-contraction if $\Phi(Tx) \leq \Phi(x)$ for any $x \in X$.

Under these definitions, the following theorem is proved in a similar way as for the Lumer-Phillips theorem on contraction semigroups.

Theorem A.(Theorem 4.1 in [2]) Let $\{T_t\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X with generator A, and let Φ be a continuous half-norm on X. Then $\{T_t\}_{t\geq 0}$ is a C_0-semigroup of Φ-contractions if and only if A is Φ-dissipative.

In case Φ is the canonical half-norm on a Banach lattice E, it is easy to see that a bounded linear operator on E is a Φ-contraction if and only if it is a positive contraction. Hence we obtain the following

Corollary B. Let $\{T_t\}_{t\geq 0}$ be a C_0-semigroup on a Banach lattice E with generator A. Then $\{T_t\}_{t\geq 0}$ is a C_0-semigroup of positive contractions if and only if A is dispersive.

The following is also an immediate consequence of Theorem A, the proof of "if" part of which is concerned with Proposition 2.1 stated below.

Corollary C. A densely defined linear operator A in a

Banach lattice E is the generator of a C_0-semigroup of positive contractions if and only if A is m-dispersive. Therefore A generates a positive C_0-semigroup if A is quasi-m-dispersive.

Whether a dispersive operator is necessarily dissipative or not is still unknown to the authors. We collect here some partial answers to this question.

Proposition 2.1. Let A be a dispersive operator in a Banach lattice E. Then the following assertions hold.

i) If the range of $\lambda - A$ is a sublattice of E for any $\lambda > 0$, then A is dissipative. In particular if A is m-dispersive, then A is m-dissipative and $\lambda R(\lambda, A)$ is a positive contraction. The same conclusion holds if A is an everywhere defined bounded operator.

ii) If E is an AM-space ([17] p.101) or an L^p space, then A is dissipative.

Proof. First we note the following facts: For any $\lambda > 0$ and $x \in D(A)$, the dispersiveness of A implies

(2.1) $\quad \|((\lambda - A)x)^+\| \geq \lambda \|x^+\|$,

(2.2) $\quad \|((\lambda - A)x)^-\| \geq \lambda \|x^-\|$.

((2.2) is obtained from (2.1) by substituting x by $-x$.) Therefore $\lambda - A$ is injective and $(\lambda - A)x \geqq 0$ implies $x \geqq 0$ for $\lambda > 0$ and $x \in D(A)$.

To prove i) we note that if $x \in D(A)$ and $\lambda > 0$,there exists a $y \in D(A)$ such that $|(\lambda - A)x| = (\lambda - A)y$ when the range of $\lambda - A$ is a sublattice of E. The inequality $(\lambda - A)(y-x), (\lambda - A)(y+x) \geqq 0$ implies $y \geqq |x|$ by the above argument. Therefore $\lambda\|x\| \leqq \lambda\|y\| \leqq \|(\lambda - A)y\| = \|(\lambda - A)x\|$ and hence A is dissipative. (The second inequality is obtained from (2.1) by substituting y for x.) The other assertions in i) is an immediate consequence of what proved above.

ii) is a direct cosequence of the inequalities (2.1) and (2.2).//

Remark 2.2. K.Sato proved an apparently more general result than Proposition 2.1. ii)([16] Theorem 5.1). But considering Bohnenblust's theorem ([8] p.137 Theorem 5), his result is essentially the same as Proposition 2.1. i).

In the rest of this section, Φ denotes the canonical half-norm on a Banach lattice E with dual Banach lattice E^*. Then as pointed out in [2], the subdifferential $\partial\Phi$ of Φ at $x \in E$ is given by

$$\partial\Phi(x) = \{f \in E^*;\ f \geqq 0\ ,\ \|f\| \leqq 1\ ,\ f(x) = \|x^+\|\},$$

which is nonvoid since Φ is continuous. On the other hand let Ψ be the convex function $x \mapsto \|x\|$ on E, and $\partial\Psi$ its subdifferential. Note that $x \mapsto \|x\|\partial\Psi(x)$ is nothing but the duality map on E.

The norm on a Banach lattice X is said to be strictly monotone if $x, y \in X$, $0 \leqq x \leqq y$ and $\|x\| = \|y\|$ imply $x = y$.

Proposition 2.3.i) $\partial\Phi(x) \subset \partial\Psi(x^+)$ holds for any $x \in E$. ii) $\partial\Phi(x) = \partial\Phi(x^+) \cap \{ f \in E^*;\ f(x^-) = 0 \}$ holds for any $x \in E$. If the norm on E^* is strictly monotone, $\partial\Phi(x) = \partial\Phi(x^+)$ for any $x \in E$ with $x^+ \neq 0$. iii) For any $x \in E$, the set

$$\partial\Phi(x) \cap \{ f \in E^*;\ f \geqq 0\ ,\ x^+ \wedge |y| = 0 \Rightarrow f(y) = 0\ (\forall y \in E)\}$$

is nonvoid. Moreover if the norm on E^* is strictly monotone and $x^+ \neq 0$, any $f \in \partial\Phi(x)$ satisfies $f(y) = 0$ if $x^+ \wedge |y| = 0$.

Proof. i) and the first part of ii) is an immediate consequence of the remarks before the proposition. To prove the second half of ii) we first show iii), for which we use the standard truncation technique. Namely we define a new functiona f_1 from $f \in E^*_+$ and $x \in E$ by putting

$$f_1(y) = \lim_{n\to\infty} f(y^+ \wedge nx^+) - \lim_{n\to\infty} f(y^- \wedge nx^+)$$

for $y \in E$. Then it readily follows that $f_1 \in E^*$ and $0 \leqq f_1 \leqq f$. Moreover, $f_1(x^+) = f(x^+)$ and $f_1(y)=0$ if $x^+ \wedge |y| = 0$. Hence f_1 belongs to the set described in iii) if $f \in \partial\Phi(x)$. The final assertion in iii) follows fro $0 \leqq f_1 \leqq f$ and $\|f_1\| = \|f\| = 1$, which holds if $f \in \partial\Phi(x)$ and $x^+ \neq 0$. The same argument also proves the second half of ii). //

Remark 2.4. The first part of iii) can be proved by a compactness argument with a minimum knowledge of Banach lattice theory. In fact, let $x \in E$ be fixed and let $y_1,\ldots,y_n$ be arbitrary finite elements of E satisfying $x^+ \wedge |y_i| = 0$ for $i = 1,\ldots,n$. Put $z := x^+ - (|y_1|+|y_2|+ \cdots +|y_n|)$. Then $z^+ = x^+$ and $z^- = |y_1|+ \cdots +|y_n|$. By ii) of Proposition 2.1, there exists an $f \in \partial\Phi(z) \subset \partial\Phi(x^+)$, which necessarily satisfies $0 \leq f(|y_i|) \leq f(z^-) = 0$ for $i = 1,\ldots,n$. This shows that the family of the w^*-compact sets of the form

$$\partial\Phi(x^+) \cap \{f \in E^*;\ f(y) = 0\},$$

indexed by $y \in E$ satisfying $x^+ \wedge |y| = 0$, has the finite intersection property. Hence the above family has nonvoid intersection, which proves the first part of iii) together with the fact $x^+ \wedge x^- = 0$ and the first part of ii).

Remark 2.5. Let Φ_0 be a continuous half-norm on a Banach space X. Then it is well-known that

$$\text{(2.3)} \quad \max\{f(y)\ ;\ f \in \partial\Phi_0(x)\} = \lim_{\varepsilon \downarrow 0} (\Phi_0(x+\varepsilon y)-\Phi_0(x))/\varepsilon$$

$$\text{(2.4)} \quad \min\{f(y)\ ;\ f \in \partial\Phi_0(x)\} = \lim_{\varepsilon \downarrow 0} (\Phi_0(x)-\Phi_0(x-\varepsilon y))/\varepsilon$$

hold for $x, y \in X$ (see e.g., Moreau [10], (10.15)). Accordingly it follows that a linear operator A in X is Φ_0-dissipative if and only if

(2.5) $\min\{f(Ax)\ ;\ f \in \partial\Phi_0(x)\} \leqq 0$ for any $x \in D(A)$

holds ([2]Theorem 3.1). Combining [2] Theorem 2.5, (2.3) and (2.4), we know that for a densely defined linear operator A in X, (2.5) is equivalent to

(2.6) $\max\{f(Ax)\ ;\ f \in \partial\Phi_0(x)\} \leqq 0$ for any $x \in D(A)$.

Thus, for a Banach lattice E with canonical half-norm Φ, given a functional $v : E \times E \to \mathbb{R}$ satisfying

(*) $\min\{f(y);\ f \in \partial\Phi(x)\} \leqq v(x,y) \leqq \max\{f(y);\ f \in \partial\Phi(x)\}$

for any $x, y \in E$, then a densely defined linear operator A in E is dispersive if and only if $v(x,Ax) \leqq 0$ for any $x \in D(A)$. This explains the former results in Phillips [14] , Hasegawa [6] and Sato [15] , since it is not difficult to see that all the functionals, which are used by the above authors to characterize the generators of positive semigroups, satisfy (*). We note that Sato was aware of essentially the same fact ([16] Remark 2.1.).

Sato's functional $\sigma(x,y)$ in [15] is of particular interest to us:

$$\sigma(x,y) = \inf\{\tau(x,(y+k)\vee(-bx));\ b \in \mathbb{R}_+\ ,\ k \perp x\}\ (x \in E_+),$$

where $\tau(u,v) = \lim_{\varepsilon\downarrow 0} (\|u+\varepsilon v\| - \|u\|)/\varepsilon$ for $u,v \in E$.

(To make things fit in with above remark, we should extend the definition of $\sigma(x,y)$ by putting $\sigma(x,y)=\sigma(x^+,y)$ for $x \in E$.) By Remark 3.3 in Sato [16], $\sigma(x,y)$ is related to the set $F(x) := \partial\Phi(x) \cap \{f \in E^*;\ f(z)=0 \text{ if } z \perp x^+\}$ appearing in

Proposition 2.3. iii):

$\sigma(x,y) = \sup\{f(y);\ f \in F(x)\}$.

Another immediate consequence of [2] Theorem 2.5 concerns the addition of operators: Let X be a Banach space with a continuous half-norm Φ_0 and let A, B be densely defined Φ_0-dissipative operators in X . Then the operator sum $A+B$ (with domain $D(A+B) = D(A)\cap D(B)$ and $(A+B)u := Au+Bu$ for $u \in D(A+B)$) is also Φ_0-dissipative.

The second half of Arendt-Chernoff-Kato [2] is devoted to treat the special case in which the underlying ordered Banach space has a positive cone with nonempty interior. The Banach lattice version of their result reads as follows.

Theorem D.(special case of [2] Theorem 5.3) Suppose that E is a Banach lattice whose positive cone has nonempty interior and A is a densely defined linear operator in E . Then A is the generator of a positive C_0-semigoup on E if and only if A satisfies the following two conditions:

(P) If $x \in D(A)\cap E_+$ and $f \in E_+^*$ such that $f(x)=0$, then $f(Ax) \geqq 0$.

(m) There exists arbitrarily large real λ such that $(\lambda - A)D(A) = E$.

The condition (P) in Theorem D was introduced by

Evans and Hanche-Olsen[4], in which they showed that a bounded linear operator A on an ordered Banach space X generates a positive semigroup if and only if A satisfies (P) provided the positive cone of X has the nearest point property. Since the positive cone of a Banach lattice has the nearest point property as remarked in the end of the first paragraph of this section, their theory is applicable to operators on Banach lattices. Nevertheless we present here a proof of Banach lattice version of their result by using the theory of dispersiveness.

Proposition 2.6. Suppose A is a bounded linear operator on a Banach lattice E. Then A generates a positive semigroup $\{e^{tA}\}_{t \geqq 0}$ if and only if A satisfies the condition (P) in Theorem D .

Proof. First we show that $A - \|A\|$ is dispersive if A satisfies (P) . Let Φ be the canonical half norm on E . Then for any $x \in E$ there exists an $f \in \partial\Phi(x)$ such that $f(y)=0$ if $y \perp x^+$(Proposition 2.3.iii)). Then

$$f((A - \|A\|)x) = f(Ax^+) - f(Ax^-) - \|A\| f(x)$$
$$\leqq f(Ax^+) - \|A\| \|x^+\| \leqq 0$$

since $f(Ax^-) \geqq 0$ by $f(x^-) = 0$ and (P) . This implies $\Phi(x - t(A - \|A\|)x) \geqq \Phi(x) - tf((A - \|A\|)x) \geqq \Phi(x)$ for any $t > 0$, hence $A - \|A\|$ is dispersive. Therefore $e^{t(A- \|A\|)} \geqq 0$ for $t \geqq 0$ by Corollary C , and hence $e^{tA} = e^{t\|A\|} e^{t(A - \|A\|)} \geqq 0$.

The necessity of (P) is obtained by differentiating the function $t \to f(e^{tA}x)$ at $t = 0$ for $x \in E_+$ .//

§3. Kato's inequality for the generator of a positive C_0-semigroup

To formulate Kato's inequality for generators of positive C_0-semigroups, we need the notion of "signum operators". Let E be a σ-order complete Banach lattice and let $u \in E$. Then there exists a unique bounded linear operator "sgn u" on E satisfying

(3.1) $\quad |(\mathrm{sgn}\, u)v| \leqq |v| \qquad v \in E ,$

(3.2) $\quad (\mathrm{sgn}\, u)v = 0 \quad \text{if} \quad u \perp v ,$

(3.3) $\quad (\mathrm{sgn}\, u)u = |u| .$

If the band projection (Schaefer [17],p.61) onto the band generated by $v \in E_+$ is denoted by P_v,

$$\mathrm{sgn}\, u = P_{u^+} - P_{u^-} .$$

For the details of the definition of signum operators, we refer the reader to Nagel-Uhlig [11].

Now a bounded linear operator B on E is said to satisfy Kato's inequality if

(K) $\quad \forall u \in E \quad B|u| \geqq (\mathrm{sgn}\, u)Bu$

holds. We give here an alternative proof of Theorem 4.1 of [11] by using the results in §2.

Proposition 3.1.(Theorem 4.1. in [11]) Let A be a bounded linear operator on a σ-order complete Banach lattice E Then $\{e^{tA}\}_{t \geq 0}$ is a positive C_0-semigroup if and only if A satisfies (K):

(K) $\forall u \in E \quad A|u| \geq (\operatorname{sgn} u)Au$.

Proof. Suppose $e^{tA} \geq 0$ for $t \geq 0$. Then for any $u \in E$ and $t > 0$

$$e^{tA}|u| \geq |e^{tA}u| \geq (\operatorname{sgn} u)e^{tA}u ,$$

hence

$$(e^{tA} - I)|u| \geq (\operatorname{sgn} u)(e^{tA} - I)u .$$

Dividing the above inequality by t and letting t tend to 0 , we obtain (K).

Conversely let A satisfy (K) and $u \in E$. Then $Au^- \geq (\operatorname{sgn} u^-)Au^-$, which implies $(I - (\operatorname{sgn} u^-))Au^- \geq 0$.

On the other hand, by Proposition 2.3. iii) , there exists an $f \in \partial\Phi(u)$ which satisfies $f(v) = 0$ if $v \perp u^+$, where Φ denotes the canonical half-norm on E . For such an f ,

$$f(Au^-) = f((I - (\operatorname{sgn} u^-))Au^-) + f((\operatorname{sgn} u^-)Au^-) \geq 0$$

holds by the above remark and $u^+ \perp (\operatorname{sgn} u^-)Au^-$. Hence

$$f((A - \|A\|)u) = f(Au^+) - \|A\| f(u^+) - f(Au^-) \leq 0 .$$

Thus $A - \|A\|$ is dispersive by Remark 2.5 , therefore $e^{tA} = e^{t\|A\|} e^{t(A - \|A\|)}$ is positive by Theorem A in §2 .//

Next we recall the notion of Yosida approximation of the generator of a C_0-semigroup. Let A be the generator of a C_0-semigroup $\{T_t\}_{t\geqq 0}$ on a Banach space X. Then $R(\lambda,A):=(\lambda-A)^{-1}$ exists for sufficiently large $\lambda\in\mathbb{R}$, and the operator $A_\lambda := \lambda\, AR(\lambda,A) = \lambda^2 R(\lambda,A) - \lambda$ is called the Yosida approximation of A. It is well known that $T_t u = \lim_{\lambda\to\infty} e^{tA_\lambda}u$ holds for any $u\in E$ and $t\geqq 0$ (Pazy[13],Theorem 5.5).

Proposition 3.2. Let $\{T_t\}_{t\geqq 0}$ be a C_0-semigroup on a σ-order complete Banach lattice E with generator A. Then the following are equivalent.

(i) $\{T_t\}_{t\geqq 0}$ is a positive semigroup.

(ii) For large $\lambda\in\mathbb{R}$, the Yosida approximation A_λ satisfies Kato's inequality:

(K) $\quad A_\lambda|u| \geqq (\operatorname{sgn} u)A_\lambda u \qquad u\in E$.

(iii) For any $h>0$, the operator $A(h):=(T_h-I)/h$ satisfies Kato's inequality:

(K) $\quad A(h)|u| \geqq (\operatorname{sgn} u)A(h)u \qquad u\in E$.

Proof. (i)$\Longrightarrow$(ii): As remarked in the introduction, $R(\lambda,A)$ is positive for large $\lambda\in\mathbb{R}$. Hence for such λ and $u\in E$,

$$\begin{aligned} A_\lambda|u| &= \lambda^2 R(\lambda,A)|u| - \lambda|u| \\ &\geqq \lambda^2|R(\lambda,A)u| - \lambda|u| \\ &\geqq \lambda^2(\operatorname{sgn} u)R(\lambda,A)u - \lambda(\operatorname{sgn} u)u \\ &= (\operatorname{sgn} u)A_\lambda u . \end{aligned}$$

(ii)$\Longrightarrow$(i):By Proposition 3.1 , the assumption (ii) implies that $e^{tA_\lambda} \geq 0$ for large λ. Hence T_t is positive since $\text{s-}\lim_{\lambda\to\infty} e^{tA_\lambda} = T_t$.

(i)$\Longrightarrow$(iii): The relation $e^{tA(h)} = \exp(tT_h/h)e^{-(t/h)}$ shows the positivity of $e^{tA(h)}$, which implies (K) for A(h) by the same argument used in the "(i)$\Rightarrow$(ii)" part.

(iii)$\Longrightarrow$(i): Again by Proposition 3.1, (iii) implies the positivity of $e^{tA(h)}$ for any $h > 0$ and $t \geq 0$. Hence $T_t \geq 0$ by the fact $T_t = \text{s-}\lim_{h\downarrow 0} e^{tA(h)}$ (Davies [3], Theorem 3.17.).//

By using Proposition 3.2 we get the following

Theorem 3.3. Let $\{T_t\}_{t\geq 0}$ be a positive C_0-semigroup on an σ-order complete Banach lattice E with generator A. Then the following weak Kato's inequality holds:

(WK) $\forall u \in D(A)\ \forall f \in D(A^+)\cap E_+^*\quad \langle |u|, A^+f\rangle \geq \langle (\operatorname{sgn} u)Au, f\rangle$,

where A^+ denotes the generator of the dual semigroup of $\{T_t\}_{t\geq 0}$ in the sense of Phillips (Pazy [13], p.39). If E is reflexive, then

(WK') $\forall u\in D(A)\ \forall f\in D(A^*)\cap E_+^*\quad \langle |u|, A^*f\rangle \geq \langle (\operatorname{sgn} u)Au, f\rangle$

holds, where A^* designates the adjoint of A.

Proof. Let Y be the closure of $D(A^*)$ in E^*. Then A^+ is the part of A^* in Y, i.e.,

$D(A^+) = \{f \in D(A^*)\ ;\ A^*f \in Y\}$ and $A^+f = A^*f$ for $f \in D(A^+)$. Therefore it is easy to see that the Yosida approximation $(A^+)_\lambda$ of A^+ has the following simple relation with the Yosida approximation of A :

$$(A^+)_\lambda = (A_\lambda)^*|_Y \qquad (\text{for large } \lambda \in \mathbb{R})$$

Therefore Proposition 3.2 implies

$$(3.4)\ \forall u \in E \quad \forall f \in Y \cap E_+^* \quad <|u|,(A^+)_\lambda f> \geq <(\mathrm{sgn}\ u)A_\lambda u,f>$$

for sufficiently large λ. If $u \in D(A)$ and $f \in D(A^+)$, then $\lim\limits_{\lambda\to\infty} A_\lambda u = Au$ and $\lim\limits_{\lambda\to\infty} (A^+)_\lambda f = A^+ f = A^* f$ hold. Thus (3.4) implies (WK) .

If E is reflexive, then (WK') holds since $A^+ = A^*$ in this case (Pazy[13],Corollary 10.6).//

The following is an immediate corollary to Theorem 3.3.

Corollary 3.4. Let $E = L^p(\mathbb{R}^n)$ $(1 \leq p < \infty$, $n \in \mathbb{N})$ and let $\{T_t\}_{t\geq 0}$ be a positive C_0-semigroup on E with generator A. Suppose A satisfies $D(A) \cap D(A^+) \supset C_0^\infty(\mathbb{R}^n)$ and A is given as a differential operator on $C_0^\infty(\mathbb{R}^n)$:

$$Au(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha u(x) \qquad (u \in C_0^\infty(\mathbb{R}^n)),$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index , $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, and $a_\alpha \in C^\infty(\mathbb{R}^n)$.

Then A satisfies the distributional,original,Kato's inequality(see Definition 4.1 below):

(K') $\forall u \in C_0^\infty(\mathbb{R}^n) \quad A|u| \geq (\mathrm{sgn}\ u)Au$.

Remark 3.5. We note that if $C_0^\infty(\mathbb{R}^n)$ is a core of A , the assumption $D(A^+) \supset C_0^\infty(\mathbb{R}^n)$ in Proposition 3.4 follows from the rest of the assumptions.

Remark 3.6. For a σ-order complete complex Banach lattice E, the generator A of a positive C_0-semigroup $\{T_t\}_{t\geq 0}$ on E satisfies the following inequality:

$$\forall u \in D(A) \ \ \forall f \in D(A^+) \cap E_+^* \quad \langle |u|, A^+ f\rangle \geq \langle \mathrm{Re}((\mathrm{sgn}\ u)Au), f\rangle \ .$$

The proof is quite similar to that of Theorem 3.3.

Another application of the Yosida approximation gives the following

Proposition 3.7. Let $\{T_t\}_{t\geq 0}$ be a positive C_0-semigroup on a Banach lattice E with generator A. Then for any $u \in D(A) \cap E_+$, $(Au)^-$ belongs to the band generated by u.

Proof. For large enough $\lambda \in \mathbb{R}$, $\lambda > 0$, let A_λ be the Yosida approximation of A. Then $A_\lambda + \lambda = \lambda^2 R(\lambda, A) \geq 0$. Suppose $u \in D(A) \cap E_+$ and $v \in E$ such that $u \perp v$. Then $A_\lambda u + \lambda u \geq 0$ implies $(A_\lambda u)^- \leq \lambda u$, hence

$$0 \leq (A_\lambda u)^- \wedge |v| \leq (\lambda u) \wedge |v| = 0 \ .$$

Together with the fact $\lim_{\lambda\to\infty} A_\lambda u = Au$, we obtain $(Au)^- \perp v$, which concludes the proof. //

Remark 3.8. $A_\lambda + \mu$ may be positive for some $\mu < \lambda$. See [9] and [11].

§4. Kato's inequality for differential operators

Let $\mathbb{N}$ be the set of all positive integers. In this section we shall use the multi-index notation:

$$\alpha = (\alpha_1, \cdots\cdots, \alpha_n) \quad \text{with} \quad |\alpha| = \sum_{i=1}^{n} \alpha_i, \qquad \alpha_i \in \mathbb{N} \cup \{0\} \ ;$$

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}, \quad D_i = \partial/\partial x_i \quad (1 \leq i \leq n).$$

We consider the formal differential operator of order m

$$A = \sum_{|\alpha| \leq m} a_\alpha(x)\, D^\alpha , \tag{4.1}$$

where $a_\alpha(x)$'s are assumed to be real valued functions in $C^\infty(\mathbb{R}^n)$.

Definition 4.1. A formal differential operator A of the form (4.1) is said to satisfy Kato's inequality if for any $u \in C_0^\infty(\mathbb{R}^n)$,the inequality

$$A\,|u| \geq (\operatorname{sgn} u)\, A\, u \tag{4.2}$$

holds in the sense of distribution, i.e., for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$,

$$\int_{\mathbb{R}^n} A^*\varphi(x)\,|u(x)|\; dx \geq \int_{\mathbb{R}^n} \varphi(x)(\operatorname{sgn} u(x))\, Au(x)dx$$

holds, where A^* denotes the formal adjoint of A .

In what follows we denote by $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ the space of all test functions (with the Schwartz topology) and distributions on $\mathbb{R}^n$, respectively, and consider that

$\mathcal{D}(\mathbb{R}^n)$ is contained in $\mathcal{D}'(\mathbb{R}^n)$.

First we prepare some facts about the change of variables of distributions. Although these facts seem to be definitely known, we state them with proofs for later references. For the time being, we use two $\mathbb{R}^n$'s , $\mathbb{R}^n_x$ and $\mathbb{R}^n_y$, and let $H : \mathbb{R}^n_x \to \mathbb{R}^n_y$ be a fixed non-singular linear mapping. Namely $y = H(x) := Cx$ $(x \in \mathbb{R}^n_x)$, where $C = (c_{ij})$ is an $n \times n$ matrix with real entries and $\det H = \det C \neq 0$. Then H induces an isomorphism

$$\varphi \in \mathcal{D}(\mathbb{R}^n_y) \quad \mapsto \quad \varphi \circ H \in \mathcal{D}(\mathbb{R}^n_x) .$$

For $T \in \mathcal{D}'(\mathbb{R}^n_x)$ define $\tilde{T} \in \mathcal{D}'(\mathbb{R}^n_y)$ by

$$(4.3) \qquad \langle \tilde{T},\varphi \rangle := \langle T,\varphi\circ H \rangle \quad \text{for} \quad \varphi \in \mathcal{D}(\mathbb{R}^n_y) .$$

Under these notations we have the following

Lemma 4.2. The mapping $T \mapsto \tilde{T}$ is a linear isomorphism from $\mathcal{D}'(\mathbb{R}^n_x)$ to $\mathcal{D}'(\mathbb{R}^n_y)$ satisfying

(a) $(\partial T/\partial x_i)^{\sim} = \sum_{j=1}^{n} c_{ji}\, \partial\tilde{T}/\partial y_j$,

(b) $(aT)^{\sim} = (a\circ H^{-1})\tilde{T}$ for $a \in C^\infty(\mathbb{R}^n_x)$,

(c) $\tilde{T} \geq 0$ if and only if $T \geq 0$.

If in particular $T \in L^1_{loc}(\mathbb{R}^n_x)$, then $\tilde{T} \in L^1_{loc}(\mathbb{R}^n_y)$ and is given by $\tilde{T} = |\det H|^{-1} (T\circ H^{-1})$.

Proof. It is obvious that the mapping $T \mapsto \tilde{T}$ becomes a linear isomorphism with property (c). Let $T \in \mathcal{D}'(\mathbb{R}^n_x)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n_y)$. Then we have

$$\langle(\partial T/\partial x_i)^{\sim},\varphi \rangle = \langle\partial T/\partial x_i, \varphi\circ H\rangle = - \langle T,(\partial/\partial x_i)(\varphi\circ H)\rangle$$

$$= - \langle T, (\sum_{j=1}^{n} c_{ji}\partial\varphi/\partial y_j)\circ H \rangle$$
$$= - \langle \tilde{T}, \sum_{j=1}^{n} c_{ji}\partial\varphi/\partial y_j \rangle$$
$$= \langle \sum_{j=1}^{n} c_{ji}\partial\tilde{T}/\partial y_j \, , \varphi\rangle .$$

This is nothing but (a). Furthermore, let $a \in C^\infty(\mathbb{R}^n_x)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n_y)$. Then we obtain (b):

$$\langle (aT)^\sim, \varphi\rangle = \langle aT, \varphi\circ H\rangle = \langle T, ((a\circ H^{-1})\varphi)\circ H\rangle$$
$$= \langle \tilde{T}, (a\circ H^{-1})\varphi\rangle$$
$$= \langle (a\circ H^{-1})\tilde{T}, \varphi\rangle .$$

The last assertion is clear. //

The formal differential operator A of the form (4.1) is considered to be a linear operator from $\mathcal{D}'(\mathbb{R}^n_x)$ into itself. The effect of the change of variables on A is given by the following

Corollary 4.3. Let A be the formal differential operator of the form (4.1), and let $T \mapsto \tilde{T}$ be the mapping defined before Lemma 4.2. Then the linear operator

$$\tilde{T} \in \mathcal{D}'(\mathbb{R}^n_y) \mapsto (AT)^\sim \in \mathcal{D}'(\mathbb{R}^n_y) \quad (T \in \mathcal{D}'(\mathbb{R}^n_x))$$

is given by the action of the following formal differential operator

$$(4.4) \qquad \tilde{A} := \sum_{|\alpha|\le m} (a_\alpha\circ H^{-1}) \prod_{i=1}^{n} (\sum_{j=1}^{n} c_{ji}\, \partial/\partial y_j)^{\alpha_i} ,$$

i.e., $(AT)^\sim = \tilde{A}\tilde{T}$ holds for $T \in \mathcal{D}'(\mathbb{R}^n_x)$.

Proof. Repeated applications of Lemma 4.2 (a) and (b) prove the assertion. //

Moreover we have

Proposition 4.4. Let A be the formal differential operator of the form (4.1) and let $\tilde{A}$ be the one given by (4.4), which is obtained from A by the change of variables $y = H(x) = Cx$, $C = (c_{ij})(\det C \neq 0)$. Then A satisfies the Kato's inequality if and only if $\tilde{A}$ does.

Proof. By using Lemma 4.2, we can show that for $u \in \mathcal{D}(\mathbb{R}^n_x)$

$$(A|u| - (\operatorname{sgn} u)\, Au)^{\sim}$$
$$= |\det H|^{-1} [\tilde{A}\, |u\circ H^{-1}| - (\operatorname{sgn}(u\circ H^{-1}))\tilde{A}(u\circ H^{-1})] .$$

Hence the assertion follows from Lemma 4.2 (c) . //

The main theorem in this section is stated as follows.

Theorem 4.5. Let a_α be a real-valued function in $C^\infty(\mathbb{R}^n)$ for every α with $|\alpha| \leqq m$. If the formal differential operator $A = \sum_{|\alpha|\leqq m} a_\alpha(x)D^\alpha$ satisfies Kato's inequality, then the order m is at most 2. Furthermore, $A_1 := \sum_{|\alpha|=2} a_\alpha(x)D^\alpha$ must be elliptic (including degenerate cases).

Proof. It suffices by translation to show that $a_\alpha(0) = 0$ for any α with $|\alpha|\geqq 3$. Set $k := \max\{|\alpha|;\ a_\alpha(0) \neq 0\}$. We shall show that Kato's inequality can not be satisfied by A if $k \geqq 3$.

Step 1). First we consider a special case. Namely, suppose that

(4.5) $\quad a_{\alpha_0}(0) \neq 0 \quad$ for $\alpha_0 = (k,0,..,0)$ with $k \geqq 3$.

Now let u_i be a function in $\mathcal{D}(\mathbb{R})$ $(i = 1,2)$ and suppose that $u_1(t) = t$ near the origin and changes the sign only once, and that $u_2(t) = 1$ near the origin and $u_2 \geq 0$. Setting

$$u(x) := u_1(x_1) \prod_{i=2}^{n} u_2(x_i) ,$$

we see that $u \in \mathcal{D}(\mathbb{R}^n_x)$ and

$$(4.6) \quad (Au)(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_1^{\alpha_1} u_1(x_1) \prod_{i=2}^{n} D_i^{\alpha_i} u_2(x_i) .$$

Next we calculate $(A|u|)(x)$. The distributional derivative of $|u|$ can be written as

$$(D^\alpha |u|)(x) = (D_1^{\alpha_1} |u_1(x_1)|) \prod_{i=2}^{n} D_i^{\alpha_i} u_2(x_i).$$

Note that the right-hand side should be regarded as a tensor product of distributions (see Schwartz [18],Chapter 4,§4,Theorem 7). In more detail,

$$D_1^{\alpha_1} |u_1(x_1)| = \begin{cases} (\operatorname{sgn} u_1(x_1)) D_1^{\alpha_1} u_1(x_1) & (\alpha_1 = 0,\ 1) \\ 2\,\delta^{(\alpha_1 - 2)}_{\{x_1 = 0\}} + (\operatorname{sgn} u_1(x_1)) D_1^{\alpha_1} u_1(x_1) & (\alpha_1 \geq 2), \end{cases}$$

where $\delta^{(\alpha_1-2)}_{\{x_1=0\}}$ is the $(\alpha_1 - 2)$-th derivative of Dirac distribution. Thus we can write down $A|u|$ as

$$(A|u|)(x) = 2 \sum_{\substack{|\alpha| \leq m \\ \alpha_1 \geq 2}} a_\alpha(x) \delta^{(\alpha_1-2)}_{\{x_1=0\}} \prod_{i=2}^{n} D_i^{\alpha_i} u_2(x_i)$$
$$+ (\operatorname{sgn} u_1(x_1)) \sum_{|\alpha| \leq m} a_\alpha(x) D_1^{\alpha_1} u_1(x_1) \prod_{i=2}^{n} D_i^{\alpha_i} u_2(x_i).$$

Therefore it follows from (4.6) that

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(4.7) $\quad (A|u| - (\mathrm{sgn}\, u) Au)(x)$

$$= 2 \sum_{\alpha_1 \geqq 2} a_\alpha(x)\delta^{(\alpha_1-2)}_{\{x_1=0\}} \prod_{i=2}^{n} D_i^{\alpha_i} u_2(x_i)$$

It remains to show that the right-hand side of (4.7) is not positive as a distribution. But, this is intuitively obvious. So, we give a proof for only one of four cases: let $k>2$ be even and $a_{\alpha_0}(0) < 0$, where α_0 is as in (4.5) . Let ψ_i be a function in $\mathcal{D}(\mathbb{R})$ ($i = 1, 2$) and suppose that $\psi_1(t) = t^{k-2}$ near the origin and $\psi_1 \geqq 0$. Next choose $b > 0$ in such a way that

$$[-b,b] \subset \{ t \in \mathbb{R} \ ; \ u_2(t) = 1 \}$$

and suppose that $\psi_2(0) = 1$, $\psi_2 \geqq 0$ and $\mathrm{supp}\, \psi_2 \subset [-b,b]$. Setting

$$\psi(x) := \psi_1(x_1) \prod_{i=2}^{n} \psi_2(x_i) \ ,$$

we see that $\psi \in \mathcal{D}(\mathbb{R}^n)$. For a multi-index α with $\alpha_1 \geqq 2$, we have

(4.8) $\quad \langle a_\alpha(x)\delta^{(\alpha_1-2)}_{\{x_1=0\}} \prod_{i=2}^{n} D_i^{\alpha_i} u_2(x_i),\ \psi(x)\rangle$

$$= (-1)^{\alpha_1} \sum_{\ell=0}^{\alpha_1-2} \binom{\alpha_1-2}{\ell} \int\cdots\int (D_1^{\ell}\psi)(0,x_2,\ldots,x_n)$$

$$\times D_1^{\alpha_1-2-\ell} a_\alpha(0,x_2,\ldots,x_n) \prod_{i=2}^{n} D_i^{\alpha_i} u_2(x_i)\, dx_2 \cdots dx_n \ .$$

If $\alpha_i > 0$ for some $i \geqq 2$, then $D_i^{\alpha_i} u_2(x_i) = 0$ on

$[-b,b]$ for such i and hence the integral on the right-hand side of (4.8) vanishes. This observation implies that $\langle A|u| - (\mathrm{sgn}\, u)\, Au \,,\, \psi \rangle/2$ is equal to

$$\sum_{\alpha_1=2}^{k} (-1)^{\alpha_1} \sum_{\ell=0}^{\alpha_1-2} \binom{\alpha_1-2}{\ell} \int\cdot\int \psi_1{}^{(\ell)}(0) \prod_{i=2}^{n} \psi_2(x_i)$$

$$\times (D_1^{\alpha_1-2-\ell} a_{(\alpha_1,0,\ldots,0)})(0,x_2,\ldots,x_n)dx_2\cdot dx_n .$$

Since $\psi_1{}^{(\ell)}(0) = 0$ ($0 \leqq \ell \leqq k-3$) and $\psi_1{}^{(k-2)} \equiv (k-2)!$, this can be simplified as

$$(k-2)!\int\cdot\cdot\int \prod_{i=2}^{n} \psi_2(x_i) a_{\alpha_0}(0,x_2,\ldots,x_n)\, dx_2 \cdots dx_n .$$

Taking a sufficiently small $b > 0$, we see that the above integral can not be positive since $a_{\alpha_0}(0) < 0$.

Step 2). Let k be the number defined in the first paragraph of the proof and suppose that $a_{\alpha_0}(0) = 0$ for $\alpha_0 = (k,0,\ldots,0)$ with $k \geqq 3$. By a linear transformation $y = H(x) := Cx$ $(x \in \mathbb{R}^n)$, A is transformed into $\tilde{A}$ in (4.4)(Corollary 4.3). The coefficient of $(\partial/\partial y_1)^k$ in $\tilde{A}$ is given by

$$\sum_{|\alpha|=k} (a_\alpha \circ H^{-1}) \prod_{i=1}^{n} c_{1i}^{\alpha_i}$$

and it does not vanish if we choose the non-singular matrix C suitably. So, we see from Step 1) and Proposition 4.4 that does not satisfy Kato's inequality if $k \geqq 3$.

Step 3). Suppose that A satisfies Kato's inequality.

Then, as was shown above, the principal part of A is given by

$$A_1 := \sum_{|\alpha|=2} a_\alpha(x)D^\alpha .$$

Since it is known that the first order terms of A satisfis Kato's equality, i.e., $(\partial/\partial x_i)|u| = (\mathrm{sgn}\ u)\partial u/\partial x_i$ as a distribution ($u \in \mathcal{D}(\mathbb{R}^n)$), we see that A_1 also satisfies Kato's inequality. We shall show that A_1 is elliptic at $x = 0$

First note that we can write

$$A_1 = \sum_{i,j=1}^{n} a_{ij}(x)D_iD_j$$

with $a_{ij}(x) = a_{ji}(x)$. So by an orthogonal transformation, A_1 at $x = 0$ is transformed into

$$\sum_{j=1}^{n} a_j(\partial/\partial y_j)^2 , \quad a_j \in \mathbb{R}\ (1 \leqq j \leqq n).$$

Then we can show that $a_j \geqq 0$ for $1 \leqq j \leqq n$ by using Proposition 4.4 and applying a similar argument as in Step 1) (equation (4.7) for transformed A_1 is again useful). Thus we are done. //

By combining Corollary 3.4, Remark 3.5 and Theorem 4.5, we obtain the following

Theorem 4.6. Let $\{T_t\}_{t\geqq 0}$ be a positive C_0-semigroup on $L^p(\mathbb{R}^n)$ ($1\leqq p<\infty$, $n \in \mathbb{N}$) with generator A . Suppose that $C_0^\infty(\mathbb{R}^n)$ is a core of A and A is given as a differential operator on $C_0^\infty(\mathbb{R}^n)$:

$$Au(x) = \sum_{|\alpha|\leqq m} a_\alpha(x)D^\alpha u(x) \qquad (u \in C_0^\infty(\mathbb{R}^n);\ m \in \mathbb{N}),$$

where $a_\alpha(x)$'s are real valued functions in $C^\infty(\mathbb{R}^n)$. Then the order of A is at most 2, and the principal part of A is elliptic including degenerate cases.

§3. m-dispersiveness of second order degenerate elliptic differential operators on $\mathbb{R}^n$

This section is concerned with the quasi-m-dispersiveness (see §2) of second order degenerate elliptic operators on $\mathbb{R}^n$. First let us consider the formal differential operator

$$\mathcal{A}u := -\sum_{j,k=1}^{n} D_j(a_{jk}(x)D_k u) + \sum_{j=1}^{n} a_j(x)D_j u + a_0(x)u ,$$

where a_{jk} , a_j and a_0 are all real-valued functions on $\mathbb{R}^n$. Basic assumptions are stated as follows.

(I) $a_{jk} \in C^2(\mathbb{R}^n)$, $a_j \in C^1(\mathbb{R}^n)$, $a_0 \in L^\infty(\mathbb{R}^n)$; the second derivatives of a_{jk} and the first order derivatives of a_j are all bounded on $\mathbb{R}^n$.

(II) For any $x \in \mathbb{R}^n$ the matrix $(a_{jk}(x))$ is positive semi-definite: for every $\xi \in \mathbb{R}^n$,

$$\sum_{j,k=1}^{n} a_{jk}(x)\,\xi_j\,\xi_k \geq 0 .$$

Let A be the maximal operator in real $L^p = L^p(\mathbb{R}^n)$ $(1<p<\infty)$ associated with $\mathcal{A}$:

$$D(A) := \{u \in L^p \; ; \; \mathcal{A}u \in L^p \text{ in the distribution sense}\},$$

$$Au := \mathcal{A}u \quad \text{for } u \in D(A) .$$

Recently Baoswang Wong-Dzung [19] has proved the following

Theorem E. Let A be the operator as above. Then $-A$ is quasi-m-dispersive in L^p $(1<p<\infty)$ and $C_0^\infty(\mathbb{R}^n)$ is a core of A .

Now we consider the possibility to relax the condition $a_0 \in L^\infty$. Let $V(x) > 0$ be a fuction in $L^p_{loc}(\mathbb{R}^n \backslash \{0\})$ and set

$$V_\varepsilon(x) := V(x)[1+\varepsilon V(x)]^{-1}, \quad \varepsilon > 0 .$$

We denote by B the maximal multiplication operator by $V(x)$ in L^p:

$$Bu(x) := V(x)u(x) \quad \text{for} \quad u \in D(B) := \{u \,;\, V(x)u \in L^p\} .$$

Then $-B$ is m-dispersive in L^p $(1<p<\infty)$ and the bounded linear operator

$$B_\varepsilon u(x) := V_\varepsilon(x)u(x) , \quad u \in L^p, \varepsilon > 0$$

is related to the Yosida approximation of $-B$ (in the sense specified in §3) by the equation $B_{1/\lambda} = -(-B)_\lambda$ $(\lambda > 0)$. Note that B_ε is also written as $B_\varepsilon = B(1+\varepsilon B)^{-1}$.

The purpose of this section is to prove the following

Thoerem 5.1. Let A and B be the operators in L^p $(1<p<\infty)$ as above. Assume that V_ε belongs to $C^1(\mathbb{R}^n)$, and there exist nonnegative constants c, a and b $(b \leq 4(p-1)^{-1})$ such that for any $\varepsilon > 0$ and $x \in \mathbb{R}^n$

$$(5.1) \quad \sum_{j,k=1}^{n} \frac{a_{jk}(x)}{V_\varepsilon(x)} D_j V_\varepsilon D_k V_\varepsilon + \frac{4}{p} \sum_{j=1}^{n} a_j(x) D_j V_\varepsilon \leq c + aV_\varepsilon(x) + b[V_\varepsilon(x)]^2 .$$

In the case of $1<p<2$ assume further that $c = 0$.

If $b < 4(p-1)^{-1}$ then $-(A+B)$ is also quasi-m-dispersive in L^p. If $b = 4(p-1)^{-1}$ then $-(A+B)$ is essentially quasi-m-dispersive on $D(A+B) := D(A) \cap D(B)$.

In any case the closure of $-(A+B)$ is the generator of a positive C_0-semigroup on L^p .

The proof of this theorem is based on the following

Lemma 5.2. Let $-A$ and $-B$ be m-dispersive operators in real $L^p(\mathbb{R}^n)$ $(1<p<\infty)$. Let D be a core of A . Assume that there exist nonnegative constants c, a and b $(b\leq 1)$ such that for any $u \in D$ and $\varepsilon > 0$,

$$\langle Au, F(B_\varepsilon u) \rangle \geq -c\,\|u\|^2 - a\,\|B_\varepsilon u\|\,\|u\| - b\,\|B_\varepsilon u\|^2$$

holds, where $B_\varepsilon = B(1+\varepsilon B)^{-1}$ and $F(B_\varepsilon u) = \|B_\varepsilon u\|^{2-p}\, B_\varepsilon u \times |B_\varepsilon u|^{p-2} \in L^q = (L^p)^*$ $((1/p)+(1/q)=1)$.

If $b<1$ then $-(A+B)$ with $D(A+B) := D(A) \cap D(B)$ is also m-dispersive in L^p. If $b=1$ then $-(A+B)$ is essentially m-dispersive on $D(A+B)$.

Proof. Since A and B are m-accretive in L^p (by Corollary C in §2 or Proposition 2.1.), it follows from Lemma 1.4 in Okazawa[12] that $A+B$ is m-accretive [resp. essentially m-accretive] if $b<1$ [resp. $b=1$] . Let C denote $A+B$ or its closure according as $b<1$ or $b=1$. Since $-(A+B)$ is dispersive (see the last paragraph of Remark 2.4) and the closure of dispersive operator is also dispersive ([2],Theorem 2.3), $-C$ is also dispersive. Thus we see that $-C$ is also m-dispersive in L^p .//

Proof of Theorem 5.1. In order to apply Lemma 5.2 , we shall show that for some constant $a^* \in \mathbb{R}$ and b, c appearing in (5.1),

$$(5.2)\quad 4\langle Au, F(B_\varepsilon u)\rangle \geq -(p-1)(c\,\|u\|^2 + a^*\|B_\varepsilon u\|\,\|u\| + b\,\|B_\varepsilon u\|^2)$$

holds for any $u \in C_0^\infty(\mathbb{R}^n)$. Since $|B_\varepsilon u(x)|^{p-2}B_\varepsilon u(x) = [V_\varepsilon(x)]^{p-1}|u(x)|^{p-2}u(x)$, we have

$$\langle Au, |B_\varepsilon u(x)|^{p-2}B_\varepsilon u(x)\rangle$$

$$= -\int_{\mathbb{R}^n} w(x)|u(x)|^{p-2}u(x) \sum_{j,k=1}^{n} D_j[a_{jk}(x)D_k u]\, dx$$

$$+ \int_{\mathbb{R}^n} w(x)|u(x)|^{p-2}u(x) \sum_{j=1}^{n} a_j(x)D_j u\, dx$$

$$+\int_{\mathbb{R}^n} a_0(x)w(x)|u(x)|^p\, dx ,$$

where we set $w(x) = [V_\varepsilon(x)]^{p-1}$. Suppose $p \geq 2$. Then w, $|u|^{p-2}u \in C^1(\mathbb{R}^n)$ and $D_j(|u|^{p-2}u) = (p-1)|u|^{p-2}D_j u$, $D_j(|u|^p) = p|u|^{p-2}uD_j u$ hold $(1 \leq j \leq n)$. Therefore by the integration by parts it follows that

$$\langle Au, |B_\varepsilon u|^{p-2}B_\varepsilon u\rangle - \int_{\mathbb{R}^n} a_0(x)w(x)|u(x)|^p\, dx$$

$$= \int_{\mathbb{R}^n} |u|^{p-2}u \sum_{j,k=1}^{n} a_{jk}(x)(D_j w)(D_k u)\, dx$$

$$+(p-1)\int_{\mathbb{R}^n} w(x)|u|^{p-2} \sum_{j,k=1}^{n} a_{jk}(x)(D_j u)(D_k u)\, dx$$

$$+\frac{1}{p}\int_{\mathbb{R}^n} w(x) \sum_{j=1}^{n} a_j(x) D_j(|u|^p)\,dx\ .$$

The sum of the first two terms on the right-hand side is not less than

$$-[4(p-1)]^{-1}\int_{\mathbb{R}^n} [w(x)]^{-1}|u|^p \sum_{j,k=1}^{n} a_{jk} D_j w D_k w\,dx\ .$$

Hence we obtain

$$\langle Au, |B_\varepsilon u|^{p-2}B_\varepsilon u\rangle \ \geq$$

$$-[4(p-1)]^{-1}\int_{\mathbb{R}^n} \frac{|u|^p}{w} \sum_{j,k=1}^{n} a_{jk}(x) D_j w D_k w\,dx$$

$$-\frac{1}{p}\int_{\mathbb{R}^n} |u|^p \sum_{j=1}^{n} a_j(x) D_j w\,dx$$

$$-\int_{\mathbb{R}^n} w(x)|u|^p\Big[\frac{1}{p}\sum_{j=1}^{n} D_j a_j - a_0(x)\Big]\,dx\ .$$

This inequality holds even if $1<p<2$. In fact, by replacing $|u(x)|^{p-2}$ in $|B_\varepsilon u|^{p-2}B_\varepsilon u$ by $[|u(x)|^2+\delta]^{(p-2)/2}$ ($\delta > 0$) and letting $\delta \downarrow 0$ after the integration by parts, we obtain the above inequality in this case. By a straightforward calculation we see from (5.1) that

$$(p-1)^{-2}\sum_{j,k=1}^{n} \frac{a_{jk}(x)}{w(x)} D_j w D_k w + \frac{4}{p(p-1)}\sum_{j=1}^{n} a_j(x) D_j w$$

$$= [V_\varepsilon(x)]^{p-2}\Big(\sum_{j,k=1}^{n} \frac{a_{jk}(x)}{V_\varepsilon(x)} D_j V_\varepsilon D_k V_\varepsilon + \frac{4}{p}\sum_{j=1}^{n} a_j(x) D_j V_\varepsilon\Big)$$

$\leq c[V_\varepsilon(x)]^{p-2} + a[V_\varepsilon(x)]^{p-1} + b[V_\varepsilon(x)]^p$,

where a is the constant appearing in (5.1).

Setting $m = \sup\{p^{-1}\sum_{j=1}^{n} D_j a_j(x) - a_0(x) \; ; \; x \in \mathbb{R}^n\}$, $a^* = a + 4m(p-1)^{-1}$ and using the Hölder inequality we obtain (5.2) for any $u \in C_0^\infty(\mathbb{R}^n)$. By Theorem E , there exists a constant M such that $-(A+M)$ is m-dispersive. For such an M , we set $a^{**} = a^* + 4M(p-1)^{-1}$. Then we have

(5.3) $4\langle (A+M)u, F(B_\varepsilon u)\rangle \geq -(p-1)(c\,||u||^2 + a^{**}\,||u||\,||B_\varepsilon u|| + b\,||B_\varepsilon u||^2$

for any $u \in C_0^\infty(\mathbb{R}^n)$. Noting that $C_0^\infty(\mathbb{R}^n)$ is a core of A+M (Theorem E.), the conclusion follows from (5.3), Lemma 5.2 and Corollary C in §2. //

Corollary 5.3. Let A and B be as in Theorem 5.1. Assume instead of (5.1) that $V(x) > 0$ is of class $C^1(\mathbb{R}^n)$ and

$$(5.4) \qquad \sum_{j,k=1}^{n} \frac{a_{jk}(x)}{V(x)} D_j V D_k V + \frac{4}{p}\sum_{j=1}^{n} a_j(x) D_j V$$

$$\leq \quad b\,[V(x) + c]^2 ,$$

where c and b $(b \leq 4(p-1)^{-1})$ are nonnegative constants. Then the conclusion of Theorem 5.2 holds.

Proof. Put $W(x) := V(x)+c$ and $W_\varepsilon(x) := W(x)[1+\varepsilon W(x)]^{-1}$

for $\varepsilon > 0$. Then by a simple calculation it can be shown that (5.4) implies (5.1) with V_ε replaced by W_ε and $a = c = 0$, whereas with b being the same as in (5.4). Therefore, by Theorem 5.1, $-(A+W) = -(A+V+c)$ is quasi-m-dispersive or essentially quasi-m-dispersive on $D(A) \cap D(W) = D(A) \cap D(V)$ according as $b < 4(p-1)^{-1}$ or $b = 4(p-1)^{-1}$. Hence the corollary is proved.//

Remark 5.4. The last paragraph of Remark 2.4 implies that the addition problem for m-dispersive operators is reduced to that of m-accretive operators. Hence there exists a possibility that the results on m-accretive operators contain the information about m-dispersive operators. In fact Okazawa[12] has implicitly shown that, for example, $\Delta - \exp(|x|^k)$ $(k \geq 1)$ is essentially m-dispersive on $C_0^\infty(\mathbb{R}^n)$ in L^p $(1<p<\infty)$, where Δ means the Laplacian in L^p. For the detail the reader is referred to [12].

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